MINIMAL VARIETIES AND QUASIVARIETIES OF SEMILATTICES WITH ONE AUTOMORPHISM

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Abstract. We describe all minimal quasivarieties and all minimal varieties of semilattices with one automorphism (considered as algebras with one binary and two unary operations).

1. Introduction

We denote by $\mathcal{S}$ the variety of $\wedge$-semilattices with one automorphism f and its inverse f^{-1}. Thus the signature of the algebras in $\mathcal{S}$ consists of one binary and two unary operation symbols. A set of equations characterizing $\mathcal{S}$ is: $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$, $x \wedge y \approx y \wedge x$, $x \wedge x \approx x$, $f(x \wedge y) \approx f(x) \wedge f(y)$, $f^{-1}(x \wedge y) \approx f^{-1}(x) \wedge f^{-1}(y)$, $f(f^{-1}(x)) \approx x$, and $f^{-1}(f(x)) \approx x$.

In this paper we are going to find all minimal subquasivarieties and all minimal subvarieties of $\mathcal{S}$. It may be considered as a continuation of the earlier paper [4], in which all subdirectly irreducible algebras in $\mathcal{S}$ were described; but the present paper is independent of that former one. For the standard terminology and basic facts from universal algebra the reader is referred to [9], for quasivarieties to [8] or [2].

Minimal varieties (quasivarieties) are the first objects of interest once we try to describe the lattice of varieties (quasivarieties) that are contained in a concrete variety (quasivariety, respectively). This is so because they are the atoms in those lattices and generated by single nontrivial algebras. However, practice shows that a concrete description of any of these lattices often ends up on providing a complete list of their atoms only or on showing that this list is uncountable (see, for example, [3], [5], [6], [11], and [12]), since in general these lattices are highly complex, if not downright idiosyncratic (see, for example, [1]). The lattice of varieties (quasivarieties) contained in $\mathcal{S}$ is uncountable; see the end of this paper. Obviously, this is far from being sufficient to claim that neither the lattice of varieties nor the lattice of quasivarieties contained in $\mathcal{S}$ admits a good mathematical description. The only proven fact, however, we can offer in this paper is that both lattices

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admit (a good) description of all their atoms even though in the case of quasivarieties infinitely many atoms are individually generated by infinite algebras.

An excellent survey of the results up to the end of the 80s on locally finite minimal varieties is [10]. A complete and satisfactory characterization in terms of generators of minimal varieties which are locally finite is contained in [7]. The literature about minimal quasivarieties which are locally finite is incomparably much more modest (see the references in [2]).

If a semilattice $\mathbf{A}$ (or an algebra from $\mathcal{S}$) has the least element, then this least element will be denoted by o_A, or just by o. Instead of saying that $\mathbf{A}$ has (or has not, resp.) the least element, we will write more briefly $o \in A$ (or $o \notin A$, resp.); in the last case we put $A \setminus \{o\} = A$.

By a flat semilattice we mean a nontrivial semilattice in which each non-least element is an atom.

By an equation $u \approx v$ we mean the ordered pair $\langle u, v \rangle$. By an inequality $u \leq v$ we mean the equation $u \wedge v \approx u$. By an equation $u \approx o$ we mean the inequality $u \leq z$, where z is a variable not occurring in u.

The set of integers is denoted by $\mathbf{Z}$, and the set of nonnegative integers by ω.

2. Minimal quasivarieties

Lemma 2.1. *Let t be a unary term of the signature of $\mathcal{S}$-algebras. Then*

1. *there exists a nonempty finite set $I \subset \mathbf{Z}$ so that $\mathcal{S} \models t(x) \approx \bigwedge_{i \in I} f^i(x)$,*
2. *t is an endomorphism on any $\mathcal{S}$-algebra $\mathbf{A}$, and*
3. *if $o \in A$, then $t(o) = o$.*

Proof. Let $\mathbf{F}$ be the one-generated free algebra in the variety of $\mathcal{S}$-algebras generated by x. Then the set $T = \{ \bigwedge_{i \in I} f^i(x) : \emptyset \neq I \subset \mathbf{Z}, |I| < \aleph_0 \}$ contains x (for $I = \{0\}$) and is closed under the meet operation. Since both f and f^{-1} are automorphisms, T is closed under f and f^{-1} as well, thus $T = F$. This proves statement (1).

To prove statement (2), let $t(x) = \bigwedge_{i \in I} f^i(x)$ for some nonempty finite set $I \subset \mathbf{Z}$. Now, $f(t(x)) \approx f\left(\bigwedge_{i \in I} f^i(x)\right) \approx \bigwedge_{i \in I} f^{i+1}(x) \approx t(f(x))$, similarly $f^{-1}(t(x)) \approx t(f^{-1}(x))$, and $t(x \wedge y) \approx \bigwedge_{i \in I} f^i(x \wedge y) \approx \bigwedge_{i \in I} \left(f^i(x) \wedge f^i(y)\right) \approx t(x) \wedge t(y)$. This proves that t is an endomorphism on any $\mathcal{S}$-algebra.

For the last statement, observe that $f^i(o) = o$ for any $i \in \mathbf{Z}$, therefore $t(o) = o$. $\square$

Lemma 2.2. *Let Q be a minimal quasivariety of $\mathcal{S}$-algebras, and $\mathbf{A} \in Q$ be a nontrivial algebra generated by $a \in A$.*

1. *If $t(a) = o \in A$ for some unary term t, then every member of Q has the least element and $Q \models t(x) \approx o$.*
2. *If $t(a) \approx s(a)$ for a pair s, t of unary terms, then $Q \models t(x) \approx s(x)$.*

Proof. Assume that $t(a) = o$ for some unary term t. Since $\mathbf{A}$ is nontrivial, it generates Q. Every element of $\mathbf{A}$ can be expressed as $r(a)$ for some unary term r, and then $t(r(a)) = r(t(a)) = r(o) = o$ by Lemma 2.1. Thus $\mathbf{A} \models t(x) = o$, that is, $\mathbf{A} \models t(x) \wedge y \approx t(x)$, and this identity must also hold in Q.

The second statement is proved similarly. $\square$

Lemma 2.3. *Let Q be a minimal quasivariety of $\mathcal{S}$-algebras, $\mathbf{A}$ be the one-generated free algebra in Q generated by $a \in A$, and*

$$K = \{\, i \in \mathbf{Z} : a \wedge f^i(a) \neq o \,\}.$$

Then either $Q \models f(x) \approx x$, or the following are true:

1. *Any nonzero element of $\mathbf{A}$ generates a nontrivial subalgebra.*
2. *If $i \in K$ and $n \geq 0$, then $a \wedge f^i(a) \wedge f^{2i}(a) \wedge \cdots \wedge f^{ni}(a) \neq o$.*
3. *There exists a unique integer $k \geq 0$ so that $K = \{\, kn : n \in \mathbf{Z} \,\}$.*
4. *For any integer $n \geq 0$, $a \wedge f^k(a) \wedge \cdots \wedge f^{nk}(a) = a \wedge f^{nk}(a)$.*
5. *$A \setminus \{o\} = \{\, f^i(a) \wedge f^{i+nk}(a) : i \in \mathbf{Z}$ and $n \geq 0 \,\}$.*
6. *If $a \wedge f^{nk}(a) \leq f^{mk}(a)$ for integers $m > n \geq 0$, then $a \leq f^k(a)$.*
7. *If $a \wedge f^{nk}(a) \leq f^{mk}(a)$ for integers $n \geq 0 > m$, then $a \geq f^k(a)$.*

Proof. Clearly, $\mathbf{A}$ is trivial if and only if $Q \models f(x) \approx x$, so we assume that A is nontrivial and prove the above statements.

For (1), take $b \in A \setminus \{o\}$. As $\mathbf{A}$ is generated by a, there exists a unary term t so that $t(a) = b$, and by Lemma 2.1, there exists $i \in \mathbf{Z}$ so that $b \leq f^i(a)$. If $b \wedge f^j(a) = b$ for all $j \in \mathbf{Z}$, then for all unary terms r, $b \wedge r(a) = b$ by Lemma 2.1 again, which implies that $b = o$. This is a contradiction, so there exists $j \in \mathbf{Z}$ so that $b \wedge f^j(a) < b$. Then $b \wedge f^{j-i}(b) \leq b \wedge f^{j-i}(f^i(a)) \leq b \wedge f^j(a) < b$, so b generates a nontrivial subalgebra.

Suppose that statement (2) does not hold, so $a \wedge f^i(a) \wedge \cdots \wedge f^{ni}(a) = o$ for some $i \in K$ and $n \geq 0$. Since $i \in K$, $a \wedge f^i(a) \neq o$, thus $n \geq 2$. Choose a counterexample where n is minimal, so $b = a \wedge f^i(a) \wedge \cdots \wedge f^{(n-1)i}(a) \neq o$. By statement (1), the subalgebra generated by b is nontrivial. Then $b \wedge f^i(b) = a \wedge f^i(a) \wedge \cdots \wedge f^{ni}(a) = o$, and from Lemma 2.2, $Q \models x \wedge f^i(x) \approx o$, which contradicts the assumption that $i \in K$.

For (3), first observe that $0 \in K$ since $a \neq o$, and for every integer i, $i \in K$ if and only if $-i \in K$ since f is an automorphism. If $K = \{0\}$, then we choose $k = 0$. If $K \neq \{0\}$, then let k be the smallest positive integer in K. From (2) and the above observation we get that $K \supseteq \{\, kn : n \in \mathbf{Z} \,\}$. Suppose that they are not equal and let $j \in K$ be the smallest positive integer not divisible by k. Since $k \in K$, $b = a \wedge f^k(a) \neq o$, and by (1) the subalgebra generated by b is nontrivial. By the choice of k and j, $j > k$ and $j - k \notin K$. Thus $a \wedge f^{j-k}(a) = o$ and $Q \models x \wedge f^{j-k}(x) \approx o$ from Lemma 2.2. In particular, $f^k(a) \wedge f^{j-k}(f^k(a)) = o$ and hence $b \wedge f^j(b) = a \wedge f^k(a) \wedge f^j(a) \wedge f^{k+j}(a) = o$. Then using Lemma 2.2 again for the subalgebra generated by b, we get that $Q \models x \wedge f^j(x) \approx o$ which contradicts that $j \in K$.

To prove (4), it is enough to show for all integers $0 \leq m \leq n$, that $a \wedge f^{nk}(a) \leq f^{mk}(a)$. Put $b = a \wedge f^k(a) \wedge \cdots \wedge f^{nk}(a)$. Then $b \wedge f^{nk}(b) = a \wedge f^k(a) \wedge \cdots \wedge f^{2nk}(a)$ and $f^{mk}(b) = f^{mk}(a) \wedge \cdots \wedge f^{(m+n)k}(a)$, thus $b \wedge f^{nk}(b) \leq f^{mk}(b)$. By (2) we know that $b \neq o$, and from (1) that b generates a nontrivial subalgebra. Therefore, from Lemma 2.2, $Q \models x \wedge f^{nk}(x) \leq f^{mk}(x)$, which implies what we wanted to show.

Now we show (5). From Lemma 2.1 we know that every element of $\mathbf{A}$ can be written as $b = \bigwedge_{i \in I} f^i(a)$ where I is a nonempty finite subset of $\mathbf{Z}$. Let i be the smallest member of I. If not all members of I are congruent to i modulo k, then $b = o$ by (3), a contradiction. Now, from (4) we get that $b = f^i(a) \wedge f^{i+nk}(a)$ where $i + nk$ is the largest member of I.

Assume that $a \wedge f^{nk}(a) \leq f^{mk}(a)$ for a pair $m > n \geq 0$ of integers. Put $b = a \wedge f^k(a) \wedge \cdots \wedge f^{(m-1)k}(a)$. From our assumption, $b \wedge f^k(b) = a \wedge f^k(a) \wedge \cdots \wedge f^{mk}(a) = b$. However, $b \neq o$ by (2), and by (1) it generates a nontrivial subalgebra. So from Lemma 2.2 we get that $Q \models x \wedge f^k(x) \approx x$. This proves statement (6).

To get (7), assume that $a \wedge f^{nk}(a) \leq f^{mk}(a)$ for a pair $n \geq 0 > m$ of integers. Then for the element $b = f^{(m+1)k}(a) \wedge \cdots \wedge f^{nk}(a)$ we get that $f^{-k}(b) \wedge b = b$. Then with a similar argument as before, $Q \models f^{-k}(x) \wedge x \approx x$, that is, $Q \models x \wedge f^k(x) \approx f^k(x)$, and thus $a \geq f^k(a)$. □

For every $k \geq 1$ we denote by $\mathbf{A}_k$ the flat algebra from $\mathcal{S}$ with underlying set $\{o, 0, \ldots, k-1\}$ where o is the least element, $f(i) = i + 1$ for $i < k - 1$ and $f(k-1) = 0$. We denote by $\mathbf{A}_\infty$ the flat algebra from $\mathcal{S}$ with underlying set $\{o\} \cup \mathbf{Z}$, where $f(i) = i + 1$ for all $i \in \mathbf{Z}$ (so that $f^{-1}(i) = i - 1$).

Denote by $\mathbf{B}_1^+$ the $\mathcal{S}$-algebra with the underlying set Z, with the usual ordering of integers and the automorphism f defined by $f(x) = x+1$. Denote by $\mathbf{B}_1^-$ the $\mathcal{S}$-algebra with the same underlying set and the same ordering, but $f(x) = x - 1$.

For every $k \geq 2$ we set $\overline{k} = \{0, 1, \ldots, k-1\}$ and denote by $\mathbf{B}_k^+$ the $\mathcal{S}$-algebra with the underlying set $(Z \times \overline{k}) \cup \{o\}$, where o is the least element, $\langle x, i\rangle \leq \langle y, j\rangle$ iff $x \leq y$ and $i = j$, $f(o) = o$, $f(\langle x, i\rangle) = \langle x, i+1\rangle$ if $i < k-1$ and $f(\langle x, k-1\rangle) = \langle x+1, 0\rangle$. Denote by $\mathbf{B}_k^-$ the $\mathcal{S}$-algebra with the same underlying set and the same ordering, but with f defined as follows: $f(o) = o$, $f(\langle x, i\rangle) = \langle x, i+1\rangle$ if $i < k-1$ and $f(\langle x, k-1\rangle) = \langle x-1, 0\rangle$.

Let $\mathbf{C}_1$ be the $\mathcal{S}$-algebra with the underlying set $\{\, \langle i, j\rangle \in \mathbf{Z} \times \mathbf{Z} : i \leq j \,\}$ with $f(\langle i, j\rangle) = \langle i+1, j+1\rangle$ and $\langle i, j\rangle \wedge \langle u, v\rangle = \langle \min(i, u), \max(j, v)\rangle$.

For an integer $k \geq 2$ denote by $\mathbf{C}_k$ the $\mathcal{S}$-algebra with underlying set $\{\, \langle i, j, m\rangle \in \mathbf{Z} \times \mathbf{Z} \times \overline{k} : i \leq j \,\} \cup \{o\}$, where o is the least element, $\langle i, j, m\rangle \leq \langle u, v, w\rangle$ iff $i \leq u$, $j \geq v$ and $m = w$, and $f(o) = o$, $f(\langle i, j, m\rangle) = \langle i, j, m+1\rangle$ if $m < k-1$ and $f(\langle i, j, k-1\rangle) = \langle i+1, j+1, 0\rangle$.

Lemma 2.4. *Let Q be a minimal quasivariety of $\mathcal{S}$-algebras, and $\mathbf{A}$ be the one-generated free algebra in Q. Then either $\mathbf{A}$ is trivial and Q is generated* *by* $\mathbf{A}_1$*, or* $\mathbf{A}$ *is nontrivial and isomorphic to* $\mathbf{A}_\infty$*,* $\mathbf{A}_k$ $(k \geq 2)$*,* $\mathbf{B}_k^+$*,* $\mathbf{B}_k^-$ *or* $\mathbf{C}_k$ *for some integer* $k \geq 1$*.*

Proof. Let $\mathbf{A}$ be the one-generated free algebra generated by $a \in A$ in a minimal quasivariety Q of $\mathcal{S}$-algebras. If $Q \models f(x) \approx x$, then $A = \{a\}$, and Q is generated by $\mathbf{A}_1$ (which is isomorphic to a subalgebra of any nontrivial member of Q). So assume that $\mathbf{A}$ is nontrivial, that is $f(a) \neq a$ and so $Q \not\models f(x) \approx x$. Thus the statements of Lemma 2.3 are satisfied.

Let k be the unique integer satisfying statement (3) of Lemma 2.3. If $k = 0$, then for any $i \neq 0$, $a \wedge f^i(a) = o$. Define $\varphi : \mathbf{A}_\infty \to \mathbf{A}$ as $\varphi(o) = o$ and $\varphi(i) = f^i(a)$ for $i \in \mathbf{Z}$. Clearly, φ is a homomorphism, and we need to show that φ is bijective. By statement (5), every element of $A \setminus \{o\}$ equals to $f^i(a) \wedge f^{i+n\cdot 0}(a) = f^i(a)$ for some $i \in \mathbf{Z}$, thus φ is surjective. On the other hand, if $f^i(a) = f^j(a)$ then $i - j \in K$, so $i = j$, which proves that φ is injective, and thus $\mathbf{A} \cong \mathbf{A}_\infty$.

In the rest of the proof we consider the case when $k \geq 1$. First assume that $a = f^k(a)$. Define $\varphi : \mathbf{A}_k \to \mathbf{A}$ as $\varphi(o) = o$ and $\varphi(i) = f^i(a)$ for $0 \leq i < k$. Clearly φ is a homomorphism, as $f^k(a) = a$, and we need to show that that φ is bijective. By statement (5), every element of $A \setminus \{o\}$ equals to $f^i(a) \wedge f^{i+nk}(a) = f^i(a) = f^{(i \bmod k)}(a)$, thus φ is surjective. On the other hand, if $f^i(a) = f^j(a)$, then $i - j \in K$, so $i \equiv j \mod k$, and therefore f is injective. This proves, that $\mathbf{A} \cong \mathbf{A}_k$. Note, that $k = 1$ is impossible in this case, because we have assumed that $\mathbf{A}$ is nontrivial.

Now consider the case when $a < f^k(a)$. If $k = 1$, then by statements (3) and (4), $\mathbf{A}$ has no least element. Define $\varphi : \mathbf{B}_1^+ \to \mathbf{A}$ as $\varphi(i) = f^i(a)$ for $i \in \mathbf{Z}$. Clearly, $f(\varphi(i)) = f^{i+1}(a) = \varphi(i+1) = \varphi(f(i))$. Since $a \leq f(a)$, for any pair $i < j$ of integers we have $f^i(a) \wedge f^j(a) = f^i(a \wedge f^{j-i}(a)) = f^i(a)$, which proves that φ is a homomorphism. From statement (5) we get again that φ is surjective. If $f^i(a) = f^j(a)$ for a pair $i < j$ of integers, then $a = f^{j-i}(a)$, and therefore, by statement (4), $a = a \wedge f^{j-i}(a) = a \wedge f(a) \wedge \cdots \wedge f^{j-i}(a) \leq a \wedge f(a) \leq a$, and consequently $a = f(a)$. This contradicts our assumption that $a \neq f(a)$. So φ is injective and $\mathbf{A} \cong \mathbf{B}_1^+$.

If $a < f^k(a)$ and $k \geq 2$, then, by statement (3), $\mathbf{A}$ has the least element. Define $\varphi : \mathbf{B}_k^+ \to \mathbf{A}$ as $\varphi(o) = o$ and $\varphi(\langle i, j \rangle) = f^{ik+j}(a)$ for any $i \in \mathbf{Z}$ and $0 \leq j < k$. Once again, it is clear that φ is compatible with f. From statement (3) we get that $\varphi(\langle i, j \rangle) \wedge \varphi(\langle u, v \rangle) = o$ whenever $j \neq v$. On the other hand, from $a \leq f^k(a)$ and statement (4) we get again, that $f^i(a) \wedge f^{i+nk}(a) = f^i(a)$ for any $n \geq 0$. Thus φ is a homomorphism. Similarly to the previous case, we can easily see that φ is bijective, and therefore $\mathbf{A} \cong \mathbf{B}_k^+$.

If $a > f^k(a)$, then a proof analogous to the one above shows that $\mathbf{A} \cong \mathbf{B}_k^-$.

The only remaining case is when a and $f^k(a)$ are incomparable. We consider the case when $k = 1$. By statement (3), $\mathbf{A}$ has no least element. Define $\varphi : \mathbf{C}_1 \to \mathbf{A}$ as $\varphi(\langle i, j \rangle) = f^i(a) \wedge f^j(a)$. Clearly φ is compatible with f. For $i \leq j$ and $u \leq v$, we get that $f^i(a) \wedge f^j(a) \wedge f^u(a) \wedge f^v(a) =$

$f^{\min(i,u)}(a) \wedge f^{\max(j,v)}(a)$ by statement (4), thus φ is compatible with the meet operation as well. We need to show that φ is a bijection. From statement (5) we get that φ is surjective. Suppose that φ is not injective, that is, $f^i(a) \wedge f^j(a) = f^u(a) \wedge f^v(a)$ for integers $i \leq j$ and $u \leq v$ with either $i \neq u$ or $j \neq v$. Without loss of generality, we may assume that $i < u$ or $j < v$. If $j < v$, then $a \wedge f^{j-i}(a) \leq f^{v-i}(a)$, and from statement (6) we get the contradiction that $a \leq f(a)$. If $i < u$, then using statement (7) instead of (6) we get another contradiction. This proves that φ is injective, and therefore it is an isomorphism.

In the case when a and $f^k(a)$ are incomparable and $k \geq 2$, a similar argument proves that $\mathbf{A} \cong \mathbf{C}_k$, and finishes the proof of the lemma. □

Theorem 2.5. *The minimal quasivarieties of $\mathcal{S}$-algebras are precisely the quasivarieties generated by one of the following algebras: $\mathbf{A}_\infty$, $\mathbf{A}_k$, $\mathbf{B}_k^+$, $\mathbf{B}_k^-$ and $\mathbf{C}_k$ for all integers $k \geq 1$. These minimal quasivarieties are pairwise distinct.*

Proof. Let $\mathcal{A}$ be the set of algebras listed above. By Lemma 2.4, every minimal quasivariety of $\mathcal{S}$-algebras contains at least one member of $\mathcal{A}$. We need to show that every member of $\mathcal{A}$ generates a minimal quasivariety, and that these quasivarieties are pairwise distinct.

If $\mathbf{A} \in \mathcal{A}$ does not generate a minimal quasivariety, then the quasivariety $\mathrm{Q}(\mathbf{A})$ generated by $\mathbf{A}$ contains a proper subquasivariety. Since every quasivariety contains a minimal quasivariety (see [8]), this subquasivariety can be taken to be minimal, and then by Lemma 2.4, $\mathrm{Q}(\mathbf{A})$ contains another algebra $\mathbf{B} \in \mathcal{A} \setminus \{\mathbf{A}\}$. So to finish the proof it is enough to show that the quasivarieties generated by different members of $\mathcal{A}$ are pairwise incomparable (under inclusion).

The algebra $\mathbf{A}_\infty$ satisfies the equation $x \wedge f^i(x) \approx o$ (or rather, the equation $x \wedge f^i(x) \wedge y \approx x \wedge f^i(x)$) for all $i \neq 0$. No other member of $\mathcal{A}$ does so, thus $\mathrm{Q}(\mathbf{A}_\infty) \cap \mathcal{A} = \{\mathbf{A}_\infty\}$, and therefore $\mathbf{A}_\infty$ generates a minimal quasivariety.

The equation $x \approx f(x)$ is satisfied by $\mathbf{A}_1$ and by no other member of $\mathcal{A}$, so $\mathbf{A}_1$ generates a minimal quasivariety. The equation $x \leq f(x)$ is satisfied exactly by $\mathbf{A}_1$ and $\mathbf{B}_1^+$. However, $\mathbf{B}_1^+$ also satisfies the quasiequation $x \approx f(x) \to y \approx z$, while $\mathbf{A}_1$ does not, therefore $\mathbf{B}_1^+$ generates a minimal quasivariety. Similarly, $\mathbf{B}_1^-$ generates a minimal quasivariety.

The algebra $\mathbf{C}_1$ satisfies the quasiequations $x \wedge f(x) \leq f^2(x) \to y \approx z$ and $f(x) \wedge f^2(x) \leq x \to y \approx z$, while no other member of $\mathcal{A}$ does so, thus $\mathbf{C}_1$ generates a minimal quasivariety.

Fix an integer $k \geq 2$. The algebra $\mathbf{A}_k$ satisfies the equations $x \wedge f^i(x) \approx o$ for every $1 \leq i < k$ and $x \approx f^k(x)$, but no other member of $\mathcal{A}$ satisfies them all, thus $\mathbf{A}_k$ generates a minimal quasivariety.

The algebra $\mathbf{B}_k^+$ satisfies the equations $x \wedge f^i(x) \approx o$ for every $1 \leq i < k$ and $x \leq f^k(x)$. The only member of $\mathcal{A}$ other than $\mathbf{B}_k^+$ that satisfies these equations is $\mathbf{A}_k$. However, $\mathbf{B}_k^+$ also satisfies the quasiequation $x \approx f^k(x) \to$

$x \le y$ while $\mathbf{A}_k$ does not, therefore $\mathbf{B}_k^+$ (and similarly $\mathbf{B}_k^-$) generates a minimal quasivariety.

The algebra $\mathbf{C}_k$ satisfies the equations $x \wedge f^i(x) \approx o$ for every $1 \le i < k$ and the quasiequations $x \wedge f^k(x) \le f^{2k}(x) \to x \le y$ and $x \wedge f^k(x) \le f^{-k}(x) \to x \le y$. If the two quasiequations are satisfied by an algebra $\mathbf{A} \in \mathcal{A}$, then for all $x \in A \setminus \{o\}$, $x \wedge f^k(x) \ne o$. But if $\mathbf{A} \models x \wedge f^i(x) \approx o$ for every $1 \le i < k$, then we must have $\mathbf{A} \in \{\mathbf{A}_k, \mathbf{B}_k^+, \mathbf{B}_k^-, \mathbf{C}_k\}$. However, among these only $\mathbf{C}_k$ satisfies the two quasiequations, so $\mathbf{C}_k$ generates a minimal quasivariety. □

3. Minimal varieties

Theorem 3.1. *The minimal quasivarieties* $\mathrm{Q}(\mathbf{A}_n)$ $(n \ge 1)$ *and* $\mathrm{Q}(\mathbf{A}_\infty)$ *are varieties. There are no other minimal subvarieties of* $\mathcal{S}$.

Proof. First we are going to prove that $\mathrm{Q}(\mathbf{A}_n)$ $(n \ge 1)$ and $\mathrm{Q}(\mathbf{A}_\infty)$ are varieties. For $n = 1$ we have the variety of semilattices. So, let $n > 1$ be an integer, or $n = \infty$.

We need to prove that an arbitrary nontrivial algebra $\mathbf{A} \in \mathrm{HSP}(\mathbf{A}_n)$ is isomorphic to a subdirect power of $\mathbf{A}_n$. Define an equivalence r on A as follows: $\langle x, y\rangle \in r$ if and only if $y = f^i(x)$ for some integer i. One block of r is $\{o\}$, all the other blocks are sets with precisely n elements. For each block $B \ne \{o\}$ of r define a binary relation α_B on A as follows: $\langle x, y\rangle \in \alpha_B$ if and only if either $x \not\ge b$ for all $b \in B$ and $y \not\ge b$ for all $b \in B$, or else there exists an element $b \in B$ such that $x, y \ge b$.

Claim 1. Let B, C be two different blocks of r, both different from $\{o\}$. Then precisely one of the following three cases takes place: (i) every element of B is incomparable with every element of C; (ii) for every element b of B there exists precisely one element c of C that is comparable with b, and this element c is above b; (iii) for every element b of B there exists precisely one element c of C that is comparable with b, and this element c is below b. Let, for example, there exist elements $b \in B$ and $c \in C$ such that $b < c$. Put $b_i = f^i(b)$ and $c_i = f^i(c)$. Then $b_i < c_i$, since f is an automorphism. If $b_i \le c_j$ for some $c_j \ne b_i$ then $c_i \wedge c_j \ge b_i$, a contradiction since $c_i \wedge c_j = o$. If $b_i \ge c_j$ for some j then $c_j \le b_i < c_i$, $c_j = c_i$ and thus $c_i = b_i$, a contradiction.

Claim 2. Let $B \ne \{o\}$ be a block of r. Then every element z of A is above at most one element of B. Suppose that $x, y \in B$, $x < z$, $y < z$ and $x \ne y$. We have $y = f^i(x)$ for some i. The elements z and $f^i(z)$ are two different elements in the same block of r, so that $z \wedge f^i(z) = o$. But they are both above y, a contradiction.

Claim 3. Let $B \ne \{o\}$ be a block of r. Then α_B is an equivalence on A. Clearly, α_B is reflexive and symmetric. Let $\langle x, y\rangle \in \alpha_B$ and $\langle y, z\rangle \in \alpha_B$. We need to prove that $\langle x, z\rangle \in \alpha_B$. If at least one of the three elements is not above any element of B, then none of them is, and we get $\langle x, z\rangle \in \alpha_B$. So, let there exist two elements $b_1, b_2 \in B$ such that $b_1 \le x$, $b_1 \le y$, $b_2 \le y$ and $b_2 \le z$. By Claim 2 we have $b_1 = b_2$ and thus $\langle x, z\rangle \in \alpha_B$.

Claim 4. Let $B \neq \{o\}$ be a block of r. Then α_B is a congruence of A. It is clear that α_B is a congruence with respect to the operation f. Let $\langle x, y\rangle \in \alpha_B$ and $z \in A$. We need to prove that $\langle x \wedge z, y \wedge z\rangle \in \alpha_B$. This is clear if neither x nor y is above any element of B. So, let there exist an element $b \in B$ such that $b \leq x$ and $b \leq y$. If $b \leq z$ then $b \leq x \wedge z$ and $b \leq y \wedge z$, so that $\langle x \wedge z, y \wedge z\rangle \in \alpha_B$. Let $b \nleq z$. If $b' \leq x \wedge z$ for some $b' \in B$ then $b = b'$ by Claim 2 and thus $b \leq z$, a contradiction. Thus $x \wedge z$ is not above any element of B. Similarly, $y \wedge z$ is not above any element of B. We get $\langle x \wedge z, y \wedge z\rangle \in \alpha_B$.

Claim 5. Let $B \neq \{o\}$ be a block of r. Then $\mathbf{A}/\alpha_B$ is isomorphic with $\mathbf{A}_n$. This is obvious.

Claim 6. The intersection of the congruences α_B, with B running over all blocks of r different from $\{o\}$, is the identity on $\mathbf{A}$. Let x, y be two different elements of A. If $x = o$ and $y \neq o$ then x, y are separated by the congruence α_B, where B is the block of y. Let $x \neq o$ and $y \neq o$. Denote by B the block of x and by C the block of y. If $B = C$ then x, y are separated by α_B. If $B \nleq C$ then x, y are separated by α_B. If $C \nleq B$ then x, y are separated by α_C.

These claims prove that $\mathbf{A}$ is isomorphic to a subdirect power of $\mathbf{A}_n$.

We see that $\mathrm{Q}(\mathbf{A}_n)$ are varieties for $n \geq 1$, including $n = \infty$. Since they are minimal quasivarieties, they are also minimal as varieties. It remains to prove that $\mathcal{S}$ has no other minimal subvarieties. By 2.5, this will be accomplished if we show that whenever a subvariety V of $\mathcal{S}$ contains either $\mathbf{B}_k^+$ or $\mathbf{B}_k^-$ or $\mathbf{C}_k$ for some $k \geq 1$, then it contains $\mathbf{A}_1$.

If V contains an algebra $\mathbf{A} \in \{\mathbf{B}_k^+, \mathbf{B}_k^-, \mathbf{C}_k\}$ for some $k \geq 2$, then V contains the algebra $\mathbf{A}_1$ because $\mathbf{A}_1$ is isomorphic to the factor $\mathbf{A}/\beta$ where β is the congruence of $\mathbf{A}$ defined as follows: $\langle x, y\rangle \in \beta$ if and only if either $x = y = o$ or $x, y \in A\backslash\{o\}$. If V contains the algebra $\mathbf{C}_1$, then it also contains $\mathbf{B}_1^+$ as the projection $\pi : \mathbf{C}_1 \to \mathbf{B}_1^+$, $\langle i, j\rangle \mapsto i$ to the first coordinate is a surjective homomorphism.

Let V contain the algebra $\mathbf{B} = \mathbf{B}_1^+$ (the underlying set of which is the set of integers). Put $\mathbf{C} = \mathbf{B}^\omega$. Denote by P the set of all $u \in C$ for which there exist a $c \in B$ and an $n \in \omega$ such that $u(i) = c$ for all $i \geq n$. Denote by Q the set of all $v \in C$ for which there exist a $c \in B$ and an $n \in \omega$ such that $v(i) = i + c$ for all $i \geq n$. It is easy to see that P, Q and $P \cup Q$ are subalgebras of C. The equivalence on $P \cup Q$ with two blocks P and Q is a congruence of the subalgebra $P \cup Q$ and the factor by this congruence is isomorphic to $\mathbf{A}_1$.

The proof is similar in the case when $\mathbf{B}_1^- \in V$. □

Remark 3.2. The fact that for n finite the algebra $\mathbf{A}_n$ generates a minimal variety follows from the profound result established in [7] which classifies all finite algebras generating minimal varieties.

Remark 3.3. It is easy to see that for $n \geq 1$ finite, the equational theory of $\mathbf{A}_n$ is based on the equations of $\mathcal{S}$, together with the following equations:

$f^n(x) \approx x$ and $x \wedge f^i(x) \approx o$ for all $i = 1, \ldots, n-1$. The equational theory of $\mathbf{A}_\infty$ is based on the equations of $\mathcal{S}$, together with the following equations: $x \wedge f^i(x) \approx o$ for all positive integers i. Thus the varieties $\mathrm{HSP}(\mathbf{A}_n)$ are finitely based for n finite, while the variety $\mathrm{HSP}(\mathbf{A}_\infty)$ is not.

Remark 3.4. There are $2^{\aleph_0}$ subvarieties of $\mathcal{S}$. To see this, denote by V_P, for any set P of prime numbers, the variety generated by the algebras $\mathbf{A}_n$ with $n \in P$. We will show that these varieties are pairwise different. It is sufficient to prove that if a prime number q does not belong to P then $\mathbf{A}_q \notin V_P$. Suppose $\mathbf{A}_q \in V_P$. Since q is a prime number, every algebra $\mathbf{A}_n$ with $n \in P$ satisfies $x \wedge f^q(x) \approx o$. Thus V_P satisfies this equation and hence $\mathbf{A}_q$ satisfies it, a contradiction.

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References

[1] M.E. Adams, K.A. Adaricheva, W. Dziobiak, A.V. Kravchenko, *Open questions related to the problem of Birkhoff and Maltsev.* Studia Logica **78** (2004), 357–378.
[2] V.A. Gorbunov, *Algebraic Theory of Quasivarieties.* Siberian School of Algebra and Logic, Consultants Bureau, 1998.
[3] J. Ježek, *On atoms in lattices of primitive classes.* Comment. Math. Univ. Carolinae **11** (1970), 515–532.
[4] J. Ježek, *Subdirectly irreducible semilattices with an automorphism.* Semigroup Forum **43** (1991), 178–186.
[5] J. Kalicki, *The number of equationally complete classes of equations.* Indag. Math. **17** (1955), 660–662.
[6] J. Kalicki and D. Scott, *Equational completeness of abstract algebras.* Indag. Math. **17** (1955), 650–659.
[7] K. Kearnes and A. Szendrei, *A characterization of minimal locally finite varieties.* Trans. Amer. Math. Soc. **349** (1997), 1749–1768.
[8] A. I. Mal'cev, *Algebraic systems.* Nauka, Moscow 1970. (in Russian)
[9] R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties, Volume I.* Wadsworth & Brooks/Cole, Monterey, CA, 1987.
[10] A. Szendrei, *A survey on strictly simple algebras and minimal varieties.* in Universal algebra and quasigroup theory (Jadwisin, 1989), Res. Exp. Math. vol. 19, Heldermann Verlag, Berlin, 1992, 209–239.
[11] A. Shafaat, *On implicational completeness.* J. Canad. Math. Soc. **26** (1974), 761–768.
[12] A. Shafaat, *The number of proper minimal quasivarieties of groupoids.* Proc. Amer. Math. Soc. **49** (1975), 54–58.

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